Stat 155 Lecture 18 Notes

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1 Threat Strategies and Nash Bargaining in Cooperative Games

1.1 Threat strategies in games with transferable utility

Players negotiate a joint strategy and a side payment. Since they are rational, they will agree to play a Pareto optimal payoff vector. Why? Players might make threats (and counter-threats) to justify their desired payoff vectors. If an agreement is not reached, they could carry out their threats. But reaching an agreement gives higher utility, so the threats are only relevant to choosing a reasonable side payment

Since players are rational, they will play on the Pareto set, which is defined by the payoff vectors with the largest total payoff,

$$\sigma := \max_{i,j}(a_{i,j} + b_{i,j}).$$

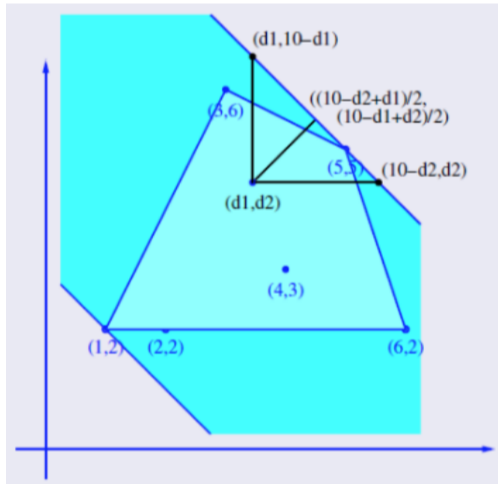
They agree on a cooperative strategy (i_0, j_0) that has $a_{i_0,j_0} + b_{i_0,j_0} = \sigma$.

The players will agree on a final payoff vector $(a^*, b^*) = (a_{i_0,j_0} - p, b_{i_0,j_0} + p)$, where p is the side payment from Player 1 to Player 2. To arrive at $(a*, b^*)$, the players agree on threat strategies $(x, y) \in \Delta_n \times \Delta_n$. We will explore how they decide on their threat strategies after we've seen how threat strategies and the final payoff vector are related. The threat strategies give a certain payoff vector, called the *disagreement point*,

$$d = (d_1, d_2) := (x^\top Ay, x^\top By).$$

Neither player will accept less than their disagreement point payoff. This defines a subset of the Pareto boundary: $(d_1, \sigma - d_1)$ to $(\sigma - d_2, d_2)$. The other details of the game are now irrelevant, so it's reasonable to choose the symmetric solution, the midpoint of this interval:

$$(a^*, b^*) = \left(\frac{\sigma - d_2 + d_1}{2}, \frac{\sigma - d_1 + d_2}{2}\right).$$



Now that we know the role of the disagreement point, we can see how the players should choose it. Player 1 wants to choose a threat strategy x to maximize $(\sigma - d_2 + d_1)/2$, and Player 2 wants to choose a threat strategy y to maximize $(\sigma - d_1 + d_2)/2$, where $(d_1, d_2) = (x^\top Ay, x^\top By)$. This is equivalent to a zero-sum game, with payoff $d_1 - d_2$ for Player 1 and payoff $d_2 - d_1$ for Player 2:

$$d_1 - d_2 = x^\top Ay - x^\top By = x^\top (A - B)y.$$

Suppose x^* and y^* are the optimal strategies for this zero-sum game with value

$$\delta = (x^*)^\top (A - B)y^*.$$

Then Player 1's best threat strategy is x^*, Player 2's best threat strategy is y^*, and the disagreement point is

$$d = (d_1, d_2) = ((x^*)^\top Ay^*, (x^*)^\top By^*).$$

The final payoff vector is

$$(a^*, b^*) = \left(\frac{\sigma + \delta}{2}, \frac{\sigma - \delta}{2}\right).$$

So Player 1 pays Player 2 $a_{i_0,j_0} - (\sigma + \delta)/2$.

Example 1.1. Consider a game with the following payoff bimatrix.

$$\begin{pmatrix} (2,2) & (6,2) & (1,2) \\ (4,3) & (3,6) & (5,5) \end{pmatrix}.$$

The payoff matrix for the zero-sum game of the disagreement points is

$$\begin{pmatrix} 0 & 4 & -1 \\ 1 & -3 & 0 \end{pmatrix}.$$

The cooperative strategy should be $(2,3)$, which gives payoff $\sigma = 10$. When we solve the zero-sum game, column 1 is dominated by column 3, so we get $x^* = (3/8, 5/8)$ and $y^* = (0, 1/8, 7/8)$. The value is $\delta = -3/8$. The disagreement point is $(3.58, 3.95)$, and the final payoff vector is $(4.8, 5.2)$. So Player 1 should pay 0.2 to Player 2.

What if two cooperative strategies are optimal? Any choice gives the same Pareto boundary. They give different disagreement points, but the value of the zero sum game $(d_1 - d_2)$ must be the same. So the payment will depend on the choice of cooperative strategy, but the final payoff vector will be the same.

1.2 Nash bargaining model for nontransferable utility games

In general, what are the ingredients of a bargaining problem? Suppose we have a compact, convex feasible set $S \subseteq \mathbb{R}^2$ and a disagreement point $d = (d_1, d_2) \in \mathbb{R}^2$. Think of the disagreement point as the utility that the players get from walking away and not playing the game. We'll assume every $x \in S$ has $x_1 \geq d_1$ and $x_2 \geq d_2$, with strict inequalities for some $x \in S$.

Definition 1.1. A *solution to a bargaining problem* is a function F that takes a feasible set S and a disagreement point d and returns an agreement point $a = (a_1, a_2) \in S$.

Here are Nash's axioms for a bargaining problem:

1. Pareto optimality: The agreement point shouldn't be dominated by another point for both players.

2. Symmetry: This is about fairness: if nothing distinguishes the players, the solution should be similarly symmetric.

3. Affine covariance: Changing the units (or a constant offset) of the utilities should not affect the outcome of bargaining.

4. Independence of irrelevant attributes: This assumes that all of the threats the players might make have been accounted for in the disagreement point.

More formally, these are

1. Pareto optimality: the only feasible payoff vector (v_1, v_2) with $v_1 \geq a_1$ and $v_2 \geq a_2$ is $(v_1, v_2) = (a_1, a_2)$.

2. Symmetry: If $(x, y) \in S \implies (y, x) \in S$ and $d_1 = d_2$, then $a_1 = a_2$.

3. Affine covariance: For any affine transformation $\psi(x_1, x_2) = (\alpha_1 x_1 - \beta_1, \alpha_2 x_2 + \beta_2)$ with $\alpha_1 > 0$ and $\alpha_2 > 0$, for any S, and for any d, $F(\psi(S), \psi(d)) = \psi(F(S, d))$.

4. Independence of irrelevant attributes: For two bargaining problems (R, d) and (S, d), if $R \subseteq S$ and $F(S, d) \in R$, then $F(R, d) = F(S, d)$.

Theorem 1.1. *There is a unique function F satisfying Nash's bargaining axioms. It is the function that takes S and d and returns the unique solution to the optimization problem*

$$\max_{x_1, x_2} (x_1 - d_1)(x_2 - d_2)$$

subject to the constraints

$$\begin{aligned} &x_1 \geq d_1 \\ &x_2 \geq d_2 \\ &(x_1, x_2) \in S. \end{aligned}$$